

On Central Root Automorphisms of Finite Generalized Octagons

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Certain techniques, developed in order to study the structure of finite projective planes which admit (non-trivial) elations, are applied to central root automorphisms of generalized octagons.

INTRODUCTION

Let Δ be the flag complex of a generalized $2m$ -gon, with $m \geq 2$, and let G be a group of special automorphisms of Δ . An automorphism of Δ is called a *central root automorphism* if it fixes every vertex at distance less than or equal to m from a prescribed vertex. This paper continues a study of the structure of the pair (Δ, G) , under the assumptions that G is finite and contains a central root automorphism distinct from the identity.

In [16] we restricted our attention to generalized hexagons (the case when $m = 3$), and here we shall consider the situation for generalized octagons ($m = 4$). This present paper is largely independent of its predecessor, although certain elementary results from [16, Sections 1–3] are required in several of the proofs. These results are reviewed (without proofs) in the next section so that this article may be read without appealing to the hexagon paper.

In order to formulate the principal results, it is necessary to introduce some notation. The reader should consult the next section for an explanation of the terminology.

Δ is a (weak) building of type $I_2(8)$, which is neither of type (D) , nor of type (E_2) when viewed as a convex subgraph of itself (in effect, this means that Δ is either the flag complex of a thick generalized octagon, or the doubled flag complex of a thick generalized quadrangle);

G is a finite group of special automorphisms of Δ ;

I is the totality of non-identity central root automorphisms in G which have centres in a preassigned type class of Δ ;

$E = \langle I \rangle$;

\mathfrak{Z} is the set of centres of the elements in I ;

\mathfrak{V} is the set of vertices in Δ which are adjacent to at least two elements in \mathfrak{Z} ;

Ξ is the subgraph of Δ which has vertices $\mathfrak{Z} \cup \mathfrak{V}$ and edges those pairs of vertices which are edges of Δ ;

$G \rightarrow \bar{G}$ ($x \mapsto \bar{x}$, for each $x \in G$) is the representation of G on Ξ ;

K is the kernel of the homomorphism $G \rightarrow \bar{G}$.

THEOREM A. *If Ξ is connected, then one of the following statements holds:*

- (I) Ξ is the empty graph;
- (IIi) ($i = 1, 2, 3, 4$ or 5) Ξ is a tree of diameter $2(i - 1)$;
- (III) Ξ is the doubled flag complex of a Moufang generalized quadrangle associated with a group of type Sp_4 , defined over some finite field \mathbb{F}_q of characteristic 2, and $\bar{E} \cong Sp_4(q)$;
- (IV) Ξ is the flag complex of a Moufang generalized octagon associated with a Ree group of type 2F_4 , defined over some finite field \mathbb{F}_q (here $q = 2^n$ with n odd), and either $\bar{E} \cong {}^2F_4(q)$, or $q = 2$ and $\bar{E} \cong {}^2F_4(2)'$.

Furthermore, if Ξ is of type III or IV, then $K = C_G(E)$ and either E is a perfect central extension of \bar{E} , or Ξ is of type III and $q = 2$.

Before turning to the disconnected case, we introduce some additional terminology. If Ξ^* is a connected component of Ξ for which statement IIIi ($i = 1, 2, \dots$, or 5) in Theorem A holds, then we say that Ξ^* is of type IIIi. The convex closure of Ξ in Δ is denoted by $\tilde{\Xi}$.

THEOREM B. *If Ξ is disconnected, then either every connected component of Ξ consists of a single vertex (that is $\mathfrak{X} = \emptyset$), or one of the following statements holds:*

- (Vi) ($i = 1$ or 2) Ξ has a connected component of type II4, each non-trivial connected component of Ξ is of type IIj, with $j = 1, 2$ or 4 , and $\tilde{\Xi}$ is a tree of diameter $2(i + 2)$;
- (VIIi) ($i = 1$ or 2) Ξ has a connected component of type II3, each non-trivial connected component of Ξ is of type IIj, with $j = 1, 2$ or 3 , and $\tilde{\Xi}$ is a tree of diameter $2(i + 2)$;
- (VIIIi) ($i = 1, 2$ or 3) Ξ has a connected component of type II2, each non-trivial connected component of Ξ is of type IIj, with $j = 1$ or 2 , and $\tilde{\Xi}$ is a tree of diameter $2(i + 1)$;
- (VIII) Ξ has a connected component of type II2, each non-trivial connected component of Ξ is of type IIj, with $j = 1$ or 2 , and $\tilde{\Xi}$ is the flag complex of a thick generalized octagon.

COROLLARY. *Suppose that G does not fix a vertex in Δ and Ξ contains a path of length 3. Then either III or IV holds.*

The proof of these theorems occupies Sections 3–5 of the paper. Background material is reviewed in Section 1. In Section 2 we study groups which are generated by pairs of central root groups. The treatment there is brief, and we restrict ourselves to establishing just those results which are needed for the subsequent sections.

The heart of the paper is contained in Sections 3 and 4. In Section 3 we study non-trivial connected components of Ξ and show that they are all trees or (weak) buildings of type $I_2(8)$. Connected components which are (weak) buildings are then classified in Section 4. In the weak case, the classification follows from some work of Ealy [2]. When the component is thick, the idea is to show that Tits's Moufang condition [13] is satisfied by exhibiting all root groups inside the stabilizer of the component. The classification then follows from Tits's [14] classification of Moufang generalized octagons. This approach goes through except when all root groups with centres in the connected component have order 2. This situation is handled separately, the simple section of the stabilizer, and with it the isomorphism type of the component, being identified by means of the centralizer of a central involution (a central root automorphism) and Parrot's characterization of ${}^2F_4(2)'$ [10].

Finally in Section 5 we show that if Ξ is disconnected, then all the non-trivial connected components of Ξ are trees of diameter at most 6. Theorems A and B are then readily deduced.

All graphs considered in this paper are undirected and without multiple edges or loops. If Δ is a graph, and if X is a vertex of Δ , then $\nu_\Delta(X)$ denotes the *valency* of X in Δ . When no confusion can arise as to which graph is under consideration, the subscript will be dropped. If Y is a vertex which is connected to X by a path in Δ , then $d_\Delta(X, Y)$ is the length of a shortest path connecting X and Y in Δ . Again, the subscript will be deleted whenever possible. If n is a positive integer, then $\Delta_n(X) = \{Y \mid d_\Delta(X, Y) = n\}$; that is, $\Delta_n(X)$ is the *sphere of radius n and centre X* . Similarly, $\Delta_{[n]}(X) = \{Y \mid d_\Delta(X, Y) \leq n\}$. We use $\text{Aut}(\Delta)$ for the group of all automorphisms of Δ .

Most group-theoretic notation is standard, and follows Gorenstein's book [4]. The places where we differ are as follows. If G is a finite group, and if $G = G'$ (the commutator subgroup), then G is called *perfect*. If A and B are subgroups of G , then $G = A \rtimes B$

means that G is a *split extension* of A by B (that is, $G = AB$ with $A \triangleleft G$ and $A \cap B = \{1\}$). A *cyclic group* of order n is denoted by Z_n and a *dihedral group* of order $2n$ by D_{2n} . If (Ω, G) is a group space and if $X \in \Omega$, then X^G is the *orbit* of X under G and G_X is the stabilizer of X in G .

A familiarity with the structure of $L_2(2^n)$, $U_3(2^n)$ and $Sz(2^{2m+1})$ is assumed, and structural properties of these groups and their natural 2-transitive permutation representations are cited without explicit reference. The reader is advised to consult [8, Kapitel II] for the linear fractional and unitary groups, and [11] for the Suzuki groups.

Let V be a vector space of even dimension $2m$ over a field \mathbb{F} . A non-empty collection \mathfrak{R} of subspaces of V is called a *partial spread* provided:

- (a) $\dim_{\mathbb{F}} U = m$ for each $U \in \mathfrak{R}$;
- (b) $U \cap W = (0)$ for all distinct $U, W \in \mathfrak{R}$.

If, in addition, \mathfrak{R} covers V , then \mathfrak{R} is called a *spread*.

An element $x \in GL(V, \mathbb{F})$ is a *shear* with *axis* X if either $x = 1$, or

$$[V, x] = X = C_V(x).$$

If \mathfrak{R} is a partial spread, and if x is a shear which leaves \mathfrak{R} invariant and has axis in \mathfrak{R} , then x is called a \mathfrak{R} -*elation* (cf. [6]).

1. BACKGROUND

Let Δ be the flag complex of a (not necessarily thick) *generalized octagon* [13]. Thus Δ is a connected bipartite graph with two types of vertices (usually called points and lines), two vertices having the same type precisely when the distance between them is even. Minimal circuits in Δ have length 16, and circuits of this length are called *apartments*. In the terminology of [12], the apartments give Δ the structure of a weak building of type $I_2(8)$.

Any two vertices X and Y are contained in an apartment, and so the distance in Δ between them is at most 8. Moreover, if $d(X, Y) < 8$, then X and Y are joined by a unique path of minimal length. We denote this path by $\langle X, Y \rangle$, and extend the notation by defining $\langle X, Y \rangle = \{X, Y\}$ when X and Y are *opposites* (that is, when $d(X, Y) = 8$). If Δ is *thick* (that is, if Δ is a building), then $\langle X, Y \rangle$ is simply the intersection of all apartments containing both X and Y .

Let Λ be a subgraph of Δ . Then Λ is called *convex* if $\langle X, Y \rangle \subseteq \Lambda$, whenever X and Y are vertices of Λ . Convex subgraphs may be conveniently classified into five distinct types.

LEMMA 1.1 [16, 1.1]. *Let Λ be a convex subgraph of Δ . Then Λ is one of the following types:*

- (A) *the empty graph;*
- (B) *a null graph with vertex set of cardinality at least 2, and with every two distinct vertices of Λ opposite in Δ ;*
- (C_d) *a tree of diameter d , with $d \leq 8$;*
- (D) *there exist two opposite vertices in Δ which are such that Λ is a non-empty union of apartments containing them;*
- (E_m) *(with $m = 2, 4$ or 8) there is a building Π of type $I_2(m)$, and $\Lambda = 8/m \Pi$.*

Concerning convex subgraphs of type (E_m), the expression $\Lambda = 8/m \Pi$ means that Λ is obtained from Π by replacing each edge by a chain of length $8/m$ [13]. Thus Λ is a (weak) building, and is thick precisely when $m = 8$.

Given an arbitrary subgraph Ω of Δ , its *convex closure* is denoted by $\bar{\Omega}$, and defined to be the intersection of all convex subgraphs of Δ which contain it. Thus $\bar{\Omega}$ is the minimal convex subgraph to contain Ω .

Now let G be a group of automorphisms of Δ . Following Tits [1, p. 45], G is called *special* if it leaves invariant each of the type classes of Δ . In other words, G is special if it is induced by a group of collineations of the generalized octagons having Δ as flag complex. Assume that G is special. Then the totality of vertices and edges of Δ which are fixed by every element of G form a convex subgraph of Δ . This subgraph is denoted by $\Delta(G)$, and is called the *fixed structure* of G . Concerning fixed structures, we have the following important result due to Tits.

LEMMA 1.2 (Tits [12, 4.1.1]). *Suppose that Δ is thick, and let $\{X, Y\}$ be an edge of Δ . Assume that G is a group of automorphisms of Δ which fixes every vertex that is adjacent to X or to Y . Then G is special, and either $G = \{1\}$, or $\Delta(G)$ is of (convex) type (C_d) with $d \geq 3$.*

We recall that a simple path of length 8 in Δ is called a *root* of Δ . Let $\Phi = (X_0, X_1, \dots, X_8)$ be a root. An automorphism of Δ fixing every vertex which is adjacent to some $X_i (i = 1, \dots, 7)$ is called a *root automorphism* for the root Φ . We define

$$G(\Phi) = \{\alpha \in G \mid \alpha \text{ is a root automorphism for } \Phi\}.$$

Then $G(\Phi)$ is a group of special automorphisms of Δ , and is called the *root group* in G for the root Φ . Clearly, $G(\Phi)$ fixes the root Φ and permutes the apartments which contain it. A straightforward application of Tits's Lemma shows that, if Δ is not "degenerate", the root group is actually semi-regular on these apartments.

LEMMA 1.3 [16, 2.1]. *If Δ is neither of type (D) , nor of type (E_2) , when viewed as a convex subgraph of itself, then $G(\Phi)$ is semi-regular on the apartments containing Φ .*

Recall that two roots in Δ are called *opposite roots* (or simply opposites), if their union is an apartment.

LEMMA 1.4 [16, 2.2]. *Suppose that G is finite and Δ satisfies the hypothesis of Lemma 1.3. Let Φ and Φ' be a pair of opposite roots in Δ , and assume that $(|G(\Phi)|, |G(\Phi')|) \neq 1$. Define $H = \langle G(\Phi), G(\Phi') \rangle$. Then $\Phi' \in \Phi^H$ and $G(\Phi')$ is conjugate in H to $G(\Phi)$.*

For the moment, assume that Δ is thick (in particular, Δ certainly satisfies the hypothesis of Lemma 1.3). If for every root Φ the root group for Φ in $\text{Aut}(\Delta)$ is transitive on the apartments containing Φ (and hence regular by Lemma 1.3), then Δ is called a *Moufang building*. In this case the incidence structures with flag complex Δ are called *Moufang generalized octagons* (this important concept is due to Tits [13]). Moufang generalized octagons may be constructed from the *BN*-pair structure of Ree groups of type 2F_4 . For details of the construction see [12, pp. 39–41]. In fact, Tits [14] has shown that all Moufang generalized octagons arise in this way. For the purpose of this paper, it is sufficient to know that finite Moufang generalized octagons are of this type (cf. [13, p. 220]).

From now on, we shall always assume that Δ satisfies the hypothesis of Lemma 1.3. Thus Δ is either the flag complex of a thick generalized octagon, or the doubled flag complex of a thick generalized quadrangle.

Let X be a vertex of Δ . An automorphism α is called a *central root automorphism with centre X* if α is a root automorphism for every root which has centre X . (The centre

of the root $\Phi = (X_0, X_1, \dots, X_8)$ is defined to be the vertex X_4 .) Clearly, Lemma 1.3 implies that if $\alpha \neq 1$, then X is the unique centre of α and the fixed structure $\Delta(\langle\alpha\rangle) = \Delta_{[4]}(X)$.

We shall need to have a reasonably precise knowledge of the fixed structure of the product of two central root automorphisms having centres of the same type (that is, their centres are distinct and an even distance apart).

LEMMA 1.5 [16, 3.1]. *Let X and X' be distinct vertices of Δ with $d(X, X') = 2d$. Furthermore let α and α' be non-trivial central root automorphisms with respective centres X and X' . Suppose that the vertex Z is fixed by $\alpha\alpha'$.*

- (i) *If $d \leq 3$, set $\langle X, X' \rangle = (X = X_0, X_1, \dots, X_{2d} = X')$. Then $d(Z, X_d) \leq \min\{8-d-1, 4+d\}$.*
- (ii) *If $d = 8$, then Z is equidistant from both X and X' , and either $d(Z, Y) \leq 3$ for some unique $Y \in \Delta_4(X) \cap \Delta_4(X')$, or Z is opposite both X and X' and $d(Z, Y) = 4$ for all $Y \in \Delta_4(X) \cap \Delta_4(X')$.*

COROLLARY 1.6. *The fixed structure $\Delta(\langle\alpha\alpha'\rangle)$ is of type (B), (C), (D) or (E_2) .*

COROLLARY 1.7 [16, 3.3]. *If $\alpha\alpha'$ is a central root automorphism, then $d = 1$ and the centre of $\alpha\alpha'$ is at distance 2 from both X and X' .*

2. GROUPS GENERATED BY TWO CENTRAL ROOT GROUPS

Given a vertex X of Δ , we define $G(X) = \{\alpha \in G \mid \alpha \text{ is a central root automorphism with centre } X\}$. Then $G(X)$ is a normal subgroup of the stabilizer G_X , and is called a *central root group*. From Lemma 1.3 it is immediate that if $\alpha \in G(X)$ with $\alpha \neq 1$, then

$$\Delta(G(X)) = \Delta_{[4]}(X) = \Delta(\langle\alpha\rangle).$$

In this section we consider the structure of groups generated by a pair of central root groups having centres of the same type.

LEMMA 2.1. *If X and X' are vertices of Δ with $d(X, X') = 2$ or 4, then $\langle G(X), G(X') \rangle = G(X) \times G(X')$.*

PROOF. Clearly, we have

$$[G(X), G(X')] \leq G(X) \cap G(X') = \{1\}.$$

LEMMA 2.2. *Let X and X' be vertices of Δ with $d(X, X') = 6$, and assume that $G(X) \neq \{1\} \neq G(X')$. Then the following hold:*

- (i) *$G(X)$, $G(X')$ and $[G(X), G(X')]$ are elementary abelian 2-groups, and $[\alpha, \beta] \neq 1$ for all $\alpha \in G(X)$ and $\beta \in G(X')$ with $\alpha \neq 1 \neq \beta$;*
- (ii) *$\langle G(X), G(X') \rangle = ([G(X), G(X')] \times G(X)) \rtimes G(X')$ is a special 2-group with centre $[G(X), G(X')]$.*

Moreover, if $\langle X, X' \rangle = (X = X_1, X_2, \dots, X_7 = X')$, then

$$\Delta_{[2]}(X_3) \cup \Delta_{[2]}(X_5) \leq \Delta([G(X), G(X')]).$$

PROOF. Let $\alpha \in G(X)$ and $\beta \in G(X')$ with $\alpha \neq 1 \neq \beta$. Then $d(X, X^\beta) = 4 = d(X', X'^\alpha)$. Therefore $[\alpha, \alpha^\beta] = 1 = [\beta, \beta^\alpha]$ by 2.1. Using these commutator identities, it is straightforward to show that $\beta^2 \beta^\alpha = \beta^{\alpha^{-1}}$. But now $d(X', X'^\alpha) = 4$ and Corollary 1.7 imply $\beta^2 = 1$, and then $\alpha^2 = 1$ follows directly. All statements in the lemma are now readily deduced.

LEMMA 2.3. *Let X and X' be opposite vertices of Δ , and assume that $G(X)$ and $G(X')$ are non-trivial elementary abelian 2-groups. Then $|G(X)| = |G(X')| = 2^n$, for some natural number n , and $\langle G(X), G(X') \rangle \cong L_2(2^n)$, $Sz(2^n)$ or D_{2m} with m odd.*

PROOF. Set $H = \langle G(X), G(X') \rangle$ and $\Omega = X^H \cup X'^H$. Then the vertices in Ω are pairwise opposites. Therefore $G(X)$ is semi-regular on $\Omega - \{X\}$ by Lemma 1.3. Similarly, $G(X')$ is semi-regular on $\Omega - \{X'\}$. Since $G(X)$ and $G(X')$ are 2-groups, Gleason's Lemma (cf. [1, p. 191]) may now be applied to conclude that H is transitive on Ω . It follows that H is the normal closure of $G(X)$ in H , and $N_H(G(X)) \cap G(X)^h = \{1\}$ for all $h \in H - N_H(G(X))$. Now the lemma follows from the main theorem in [5] and the assumption that $G(X)$ is abelian.

LEMMA 2.4. *Let (X_1, X_2, \dots, X_9) be a root in Δ , and assume that $G(X_i) \neq \{1\}$ for $i = 1, 7$ and 9 . Then $G(X_i)$ is a non-trivial elementary abelian 2-group for $i = 1, 3, 7$ and 9 . Furthermore, if $H = \langle G(X_1), G(X_9) \rangle$, then $X_9 \in X_1^H$ and $X_7 \in X_3^H$.*

PROOF. By Lemma 2.2(i) both $G(X_1)$ and $G(X_7)$ are elementary abelian 2-groups. Choose $1 \neq \alpha \in G(X_1)$. Then $d(X_7^\alpha, X_9) = 6$, and so Lemma 2.2 shows that $G(X_9)$ is also an elementary abelian 2-group. From Lemma 2.3 we now have that $X_9 \in X_1^H$, and hence $X_7 \in X_3^H$ because X_5 is fixed by H .

We conclude this section with a technical remark which will be required in the proof of Theorem 4.2.

LEMMA 2.5. *Assume that Δ is thick. Let $\Sigma = (X_0, X_1, \dots, X_{15})$ be an apartment, and assume that $G(X_{13})$ has even order. Let α be an involution in G which satisfies:*

- (a) α fixes a root $\Phi = (X_1, \dots, X_9)$;
- (b) *for some edge $\{X_i, X_{i+1}\}$ of Φ , the involution α fixes every vertex in Δ which is adjacent to X_i or X_{i+1} .*

Then $\alpha \in G(X_5)$.

PROOF. Let β be an involution in $G(X_{13})$. Then $\langle \alpha, \beta \rangle \cong D_{2m}$ for some $m \geq 3$. If m is odd, then we are finished. By way of a contradiction, assume that m is even. By Lemma 1.2, $\langle \alpha \rangle$ is semi-regular on the apartments containing Φ . In particular, X_{13} and X_{13}^α are opposites. Therefore $\beta\beta^\alpha$ has odd order by Lemma 2.3, and consequently $m \not\equiv 0 \pmod{4}$. Hence the involution in the centre of $\langle \alpha, \beta \rangle$ can be written as $z = \alpha\gamma$, where γ is a conjugate of β . The centre of γ is at distance 4 from both X_1 and X_9 , yet it must be fixed by α because α commutes with γ . Therefore γ has centre X_5 . But now $z = \alpha\gamma$ fixes every vertex which is adjacent to X_i or X_{i+1} . It also fixes X_{13} because it centralizes β . This contradicts Lemma 1.2.

3. CONNECTED COMPONENTS OF Ξ

We now begin to study the graph Ξ (described in the introduction) which is determined by the centres of the central root automorphisms in G . In this section, we shall show that all connected components of Ξ are convex in Δ (cf. Theorem 3.9).

Given a non-trivial connected component Ξ^* of Ξ , we write its vertex set as the union of type classes $\mathfrak{Z}^* \cup \mathfrak{Z}^*$, where $\mathfrak{Z}^* \subseteq \mathfrak{Z}$ and $\mathfrak{Z}^* \subseteq \mathfrak{Z}$. The subset of I consisting of those elements which have their centre in \mathfrak{Z}^* is denoted by I^* , and $E^* = \langle I^* \rangle$. We denote the stabilizer of Ξ^* in G by G^* (instead of G_{Ξ^*}). Clearly, both I^* and E^* are normal in G^* .

If Y belongs to \mathfrak{J} , then we define

$$M(Y) = \langle G(X) | X \in \Xi_1(Y) \rangle.$$

It is clear that, if $X \in \mathfrak{J}^*$ and if $Y \in \mathfrak{J}^*$, then both $G(X)$ and $M(Y)$ are subgroups of G^* .

The kernel of the restriction of G^* to Ξ^* (or equivalently to \mathfrak{J}^*) is K^* , and we let \sim be the canonical homomorphism of G^* onto G^*/K^* . Thus \tilde{G}^* is a group of automorphisms of Ξ^* .

LEMMA 3.1. *Suppose that Ξ^* is a non-trivial convex connected component of Ξ , then Ξ^* is of type (C_d) with d even, type (E_4) or type (E_8) .*

PROOF. As Ξ^* is a non-trivial and connected convex subgraph of Δ , it must be of one of the type (C_d) , (D) or (E_m) . If Ξ^* is of type (C_d) , then the extremities of Ξ^* obviously belongs \mathfrak{J}^* , and so d is even. Assume then that Ξ^* contains an apartment $\Sigma = (X_0, X_1, \dots, X_{15})$, with say $X_1 \in \mathfrak{J}^*$. Then $\nu_{\Xi^*}(X_{2i+1}) \geq 3$ for each $i \in \mathbb{Z} \mid 16\mathbb{Z}$ because $G(X_{2i-3})$ fixes both X_{2i} and X_{2i+1} , yet moves X_{2i+2} . Hence Ξ^* is of type (E_4) or of type (E_8) .

LEMMA 3.2. *If Ξ^* is a convex connected component of Ξ of type (C_6) , then $G(X)$ is an elementary abelian 2-group for each extreme vertex X of Ξ^* .*

PROOF. This is immediate from Lemma 2.2(i).

LEMMA 3.3. *If Ξ^* is a convex connected component of Ξ of type (C_8) , then there is a unique $X' \in \mathfrak{J}^*$ with $G(X') \leq Z(E^*)$. Moreover, $G(X)$ is an elementary abelian 2-group for every $X \in \mathfrak{J}^* - \{X'\}$.*

PROOF. Let X' be that unique vertex in Ξ^* which satisfies $\Xi^* \subseteq \Delta_{[4]}(X')$. Then $X' \in \mathfrak{J}^*$, and $G(X') \leq Z(E^*)$ by Lemma 2.1. Clearly, X' is the only vertex in \mathfrak{J}^* which is fixed by E^* . The final observation follows from Lemma 2.2(i) because every vertex in $\mathfrak{J}^* - \{X'\}$ is at distance 6 from another.

LEMMA 3.4. *Let $\Sigma = (X_0, X_1, \dots, X_{15})$ be an apartment in Δ . Assume that $\langle X_7, X_{11} \rangle$ is contained in a connected component Ξ^* and that $X_1 \in \mathfrak{J}$. Then $\Sigma \subseteq \Xi^*$, and $G(X_{2i+1})$ is an elementary abelian 2-group for each $i \in \mathbb{Z} \mid 16\mathbb{Z}$.*

PROOF. This is a straightforward application of Lemma 2.4 to suitably chosen roots in Σ .

LEMMA 3.5. *Let Ξ^* be a connected component of Ξ , and $\Sigma = (X_0, X_1, \dots, X_{15})$ an apartment in Δ . Assume that $X_1 \in \mathfrak{J}^*$ and that $(X_1, X_2, \dots, X_{10}) \subseteq \Xi^*$. Then either $\Sigma \subseteq \Xi^*$, or $\Sigma \cap \Xi^* = (X_0, X_1, \dots, X_{10})$.*

PROOF. From Lemma 2.4, it follows that X_0 lies in the orbit of X_{10} under $\langle G(X_1), G(X_9) \rangle$. Therefore, $(X_0, X_1, \dots, X_{10}) \subseteq \Xi^*$. If X_{12} is adjacent to a vertex in \mathfrak{J}^* , then $X_{11} \in \mathfrak{J}^*$ by Lemma 2.4 again, whence $\Sigma \subseteq \Xi^*$ by Lemma 3.4. Similarly, if X_{14} is adjacent to a vertex in \mathfrak{J}^* , then $\Sigma \subseteq \Xi^*$. So either $\Sigma \cap \Xi^* = (X_0, X_1, \dots, X_{10})$, or $\Sigma \subseteq \Xi^*$.

LEMMA 3.6. *If Ξ^* is a non-convex connected component of Ξ , then there exists an apartment Σ in Δ with Σ satisfying the assumptions of Lemma 3.5 but with $\Sigma \not\subseteq \Xi^*$.*

PROOF. As Ξ^* is assumed to be non-convex, we can find distinct $X, Y \in \mathfrak{J}^* \cup \mathfrak{J}^*$ with $\langle X, Y \rangle \not\subseteq \Xi^*$. Since Ξ^* is connected, X and Y are linked by a path in Ξ^* . Let $(X = X_1, \dots, X_{r+1} = Y)$ be such a path, chosen so that its length r is minimal. Then $r \geq 9$ because $\langle X, Y \rangle \not\subseteq \Xi^*$. Let Σ be the apartment in Δ which contains the path $(X_1, X_2, \dots, X_{10})$. Then $(X_1, X_2, \dots, X_{10}) \subseteq \Xi^*$ by construction, but $\Sigma \not\subseteq \Xi^*$ by the minimality of r . By reversing the labelling of $(X_1, X_2, \dots, X_{10})$ if necessary, it may be assumed that $X_1 \in \mathfrak{J}^*$.

PROPOSITION 3.7. *Suppose that Ξ^* is a non-trivial connected component of Ξ which is not convex of type (C_d) . Then either E^* is transitive on \mathfrak{J}^* , or Ξ^* is convex of type (E_4) and E^* has exactly two orbits on \mathfrak{J}^* . In either case, E^* is transitive on \mathfrak{J}^* and $G(X)$ is an elementary abelian 2-group for each $X \in \mathfrak{J}^*$.*

PROOF. Suppose that Ξ^* is convex. Then Lemma 3.1 and the assumption that Ξ^* is not of type (C_d) imply that Ξ^* is a (weak) building of type $I_2(8)$. Therefore every pair of vertices of Ξ^* is included in an apartment of Ξ^* . Hence Lemma 3.4 shows that $G(X)$ is an elementary abelian 2-group for every $X \in \mathfrak{J}^*$.

Let $\Sigma = (X_0, X_1, \dots, X_{15})$ be an apartment in Σ with $X_0 \in \mathfrak{J}^*$. Then applying Lemma 2.4, to sufficiently many roots in Σ , shows that the sets $\{X_0, X_2, \dots, X_{2i}, \dots, X_{14}\}$, $\{X_1, X_5, X_9, X_{13}\}$ and $\{X_3, X_7, X_{11}, X_{15}\}$ are each contained in an orbit of E^* . Hence E^* is transitive on \mathfrak{J}^* and has at most two orbits on \mathfrak{J}^* .

Suppose that Ξ^* is not of type (E_4) , and choose $X \in \mathfrak{J}^*$ to be adjacent to X_0 but distinct from X_1 and X_{15} . Then two applications of Lemma 2.4 (to suitably chosen roots) show that X lies in the orbit of X_7 as well as in the orbit of X_9 under E^* . Therefore E^* is transitive on \mathfrak{J}^* .

Now assume that Ξ^* is of type (E_4) . If $X, X' \in \mathfrak{J}^*$, then $d(X', X^\alpha)$ is even for all $\alpha \in G(X')$. This means that E^* induces a group of special automorphisms on the building of type $I_2(4)$ which underlies Ξ^* . Hence E^* has two orbits on \mathfrak{J}^* .

We may now assume that Ξ^* is non-convex. Let $\Sigma = (X_0, X_1, \dots, X_{15})$ be an apartment of the type described in Lemma 3.6. Choose $X \in \Xi_1^*(X_0) - \{X_1\}$ and $X' \in \Xi_1^*(X_{10}) - \{X_9\}$. Applying Lemma 2.4 to appropriate roots in the path $(X, X_0, X_1, \dots, X_{10}, X')$ shows that each of the groups $G(X)$, $G(X')$ and $G(X_{2i+1})$ ($i = 0, 1, 2, 3, 4$) is an elementary abelian 2-group. Since X is not a vertex in Σ , it is opposite in Δ to both X_7 and X_9 . Hence X, X_7 and X_9 lie in an E^* orbit by Lemma 2.3.

Now multiple application of Lemma 2.4 shows that the vertices X, X' and X_{2i+1} (with $i = 0, 1, \dots$ or 4) are included in a common E^* orbit. Then the transitivity of E^* on \mathfrak{J}^* follows upon showing that if $Z \in \mathfrak{J}^*$, then the orbit Z^{E^*} has non-trivial intersection with $\Pi = (X, X_0, X_1, \dots, X_{10}, X')$. We may assume that Z is not in this path. By connectivity, Z is connected to Π by a path in Ξ^* . Let $(Z = Z_1, \dots, Z_{r+1})$ be such a path, chosen to have minimal length r . So Z_{r+1} is in Π but Z_i is not in Π for all $i \leq r$. Write $r = 8k + d$, with k and d non-negative integers and $d \leq 7$. Successive application of Lemma 2.4 shows that Z_{8k+1} is in the E^* orbit of Z . Therefore we may assume that $k = 0$ and $d \geq 1$. Now extend (Z_1, \dots, Z_{r+1}) by a path in Π of length $8 - r$. This is possible because if $r = 1$, then $Z_{r+1} \in \mathfrak{J}^*$. Applying Lemma 2.4 again shows that Z^{E^*} contains a vertex from Π .

Finally, let $Y \in \mathfrak{J}^*$. Then the transitivity of E^* on \mathfrak{J}^* means that Y^{E^*} contains a vertex Y' which is adjacent to X_1 . But now appropriate use of Lemma 2.4 shows that Y' is in the E^* orbit of X_0 .

COROLLARY 3.8. *If Ξ^* is a connected component of Ξ and if $X, X' \in \mathfrak{J}^*$, then $\langle X, X' \rangle \subseteq \Xi^*$.*

PROOF. Obviously there is nothing to prove if Ξ^* is convex. Suppose that Ξ^* is non-convex. If $d(X, X') = 0, 2$ or 8 , then trivially $\langle X, X' \rangle \subseteq \Xi^*$. So we only need to consider the cases $d(X, X') = 4$ or 6 . Let $\Sigma = (X_0, X_1, \dots, X_{15})$ be an apartment of the type described in Lemma 3.6. As E^* is transitive on \mathfrak{B}^* by Proposition 3.7, we may assume that $X = X_1$.

First suppose that $d(X_1, X') = 4$, and set $\langle X_1, X' \rangle = (X_1 = X'_1, X'_2, \dots, X'_5 = X')$. If $X'_3 = X_3$, then there is nothing more to show. Assume that $X'_3 \neq X_3$. If $X'_2 = X_2$, form the root $(X'_5, X'_4, X'_3, X_2, \dots, X_7)$, whilst if $X'_2 \neq X_2$, form the root $(X'_5, X'_4, \dots, X_1, \dots, X_5)$. An application of Lemma 2.4 to these roots shows that $X'_3 \in \mathfrak{B}^*$. Hence $\langle X_1, X' \rangle \subseteq \Xi^*$.

Finally, suppose that $d(X_1, X') = 6$, and set $\langle X_1, X' \rangle = (X_1 = X'_1, X'_2, \dots, X'_7 = X')$. If $X'_3 = X_3$, then $\langle X'_3, X' \rangle \subseteq \Xi^*$ by the above paragraph, and there is nothing more to show. Assume that $X'_3 \neq X_3$. If $X'_2 = X_2$, form the root $(X_5, X_4, \dots, X'_2, \dots, X'_7)$, whilst $X'_2 \neq X_2$, form the root $(X_3, X_2, X'_1, \dots, X'_7)$. Then Lemma 2.4 applied to these roots shows that $X'_5 \in \mathfrak{B}^*$. But now $\langle X'_1, X'_5 \rangle \subseteq \Xi^*$ by the paragraph above, and hence $\langle X_1, X' \rangle \subseteq \Xi^*$ as required.

THEOREM 3.9. *If Ξ^* is a non-trivial connected component of Ξ , then Ξ^* is convex and of type (C_d) with d even, type (E_4) or type (E_8) .*

PROOF. In view of Lemma 3.1, it is sufficient to show that each connected component of Ξ is convex. Assume this to be false, and let Ξ^* be a counterexample.

Fix $\Sigma = (X_0, X_1, \dots, X_{15})$ to be an apartment of the type described in Lemma 3.6. Thus $X_0 \in \mathfrak{B}^*$ and $\Sigma \cap \Xi^* = (X_0, X_1, \dots, X_{10})$.

LEMMA 3.10. *If $Y \in \mathfrak{B}^*$, then $M(Y)$ is an elementary abelian 2-group which is partitioned by $\{G(X) \mid X \in \Xi_1^*(Y)\}$.*

PROOF. By Proposition 3.7, E^* is transitive on \mathfrak{B}^* and $G(X)$ is an elementary abelian 2-group for all $X \in \mathfrak{B}^*$. In particular, we may assume that $Y = X_0$. Then $M(X_0)$ is an elementary abelian 2-group by Lemma 2.1. Let $X, X' \in \Xi_1^*(X_0)$ be distinct, and suppose that $\alpha \in G(X)$ and $\beta \in G(X')$ with $\alpha \neq 1 \neq \beta$.

We argue that $\alpha\beta \in I^*$. Then Lemma 3.10 follows from Corollary 1.7.

By way of a contradiction, assume $\alpha\beta \notin I^*$. Choose $1 \neq \gamma \in G(X_9)$. Then $\langle \gamma, \alpha\beta \rangle \cong D_{2m}$ for some natural number m , and m is even because $\alpha\beta \notin I^*$.

First we argue that $m \not\equiv 0 \pmod{4}$. By Lemma 1.5(i), we have $X_{10}^{\alpha\beta} \neq X_{10}$. Suppose that $\alpha\beta$ fixes X_{12} . Then $d(X_9, X_9^{\alpha\beta}) = 4$ or 6 , and in either case X_{11} is in $\langle X_9, X_9^{\alpha\beta} \rangle$. But now $X_{11} \in \mathfrak{B}^*$ by Corollary 3.8, and this contradicts the choice of Σ . Therefore $X_{12}^{\alpha\beta} \neq X_{12}$, and it follows that X_9 and $X_9^{\alpha\beta}$ are opposites. Hence $\gamma\gamma^{\alpha\beta}$ has odd order by Lemma 2.3. Hence $m \not\equiv 0 \pmod{4}$ because $m/2 = o(\gamma\gamma^{\alpha\beta})$.

Let z be the involution in $Z(\langle \gamma, \alpha\beta \rangle)$. Then $z = \alpha\beta\omega$ for some conjugate ω of γ in $\langle \gamma, \alpha\beta \rangle$. Let $W \in \mathfrak{B}^*$ be the centre of ω . Then $d(W, X_{13}) = 4$, and $d(W, X_0) \leq 5$ by Lemma 1.5(i). If W is adjacent to X_0 , then $\Delta_3(X_0)$ and X_9 are contained in $\Delta(\langle z \rangle)$, and Lemma 1.2 is contradicted. (Note that Δ is certainly thick because both X_0 and X_1 have valency at least 3.) Therefore $d(W, X_0) \geq 3$, and $d(W, X_{15}) = 2$ or $d(W, X_{14}) = 3$. In either case, X_{15} belongs to $\langle W, X_1 \rangle$, and so $X_{15} \in \mathfrak{B}^*$ by Corollary 3.8. Again we contradict our choice of Σ .

LEMMA 3.11. *If $X \in \mathfrak{B}^*$, then $G(X) \cong Z_2$.*

PROOF. Set $|G(X)| = q$. Then q is a power of 2 and is independent of the choice of X . Assume that $q \neq 2$. Set $H = \langle G(X_1), G(X_9) \rangle$, and define $Z = N_H(G(X_1)) \cap N_H(G(X_9))$.

Thus $H \cong L_2(q)$ or $Sz(q)$ by Lemma 2.3, $Z \cong Z_{q-1}$ and $N_H(Z) \cong D_{2(q-1)}$. In particular, each element in Z can be written as the product of two involutions in H . Since all such involutions belong to I^* and have centres in $\Delta_4(X_5) \cap \Delta_4(X_{13})$, and as Z fixes each vertex in Σ , it follows from Corollary 1.6 that Z acts semi-regularly on $\Xi_1^*(X_2) - \{X_1, X_3\}$ and on $\Xi_1^*(X_0) - \{X_1\}$. But X_2 and X_0 have the same valency in Ξ^* by Proposition 3.7. Therefore $q = 2$ after all.

LEMMA 3.12. *If $Y \in \mathfrak{J}^*$, then $M(Y) \cong Z_2 \times Z_2$.*

PROOF. In view of the transitivity of E^* on \mathfrak{J}^* , it is sufficient to prove Lemma 3.12 for the special case $Y = X_8$. For $i = 1, 3, \dots, 9$, set

$$G(X_i) = \langle \alpha_i \rangle, \text{ where } \alpha_i^2 = 1 \neq \alpha_i.$$

Define $\alpha = \alpha_7\alpha_9$. Then $\alpha \in I^*$ and α has centre $X \in \Xi_1^*(X_8) - \{X_7, X_9\}$ by Lemma 3.10. So $\langle \alpha_1, \alpha \rangle \cong D_{2m}$, with m odd by Lemma 2.3. Therefore there exists $\delta \in \langle \alpha_1, \alpha \rangle \cap I^*$ such that δ interchanges X_1 and X . Certainly δ fixes X_5 because X_5 is fixed by both α_1 and α . Hence, δ leaves invariant the root $(X_1, X_2, \dots, X_8, X)$, and δ has centre Z lying in $\Delta_4(X_5)$ and opposite to both X_1 and X . Similarly, there exists $\delta' \in \langle \alpha_1, \alpha_9 \rangle \cap I^*$ leaving invariant the root $(X_1, X_2, \dots, X_8, X_9)$, and having centre Z' in $\Delta_4(X_5)$ and opposite to both X_1 and X_9 . Therefore $\delta\delta' \neq 1$ and $\delta\delta'$ fixes every vertex in the path (X_2, X_3, \dots, X_8) .

We argue that $d(Z, Z') = 4$ or 6 . Certainly $Z \neq Z'$ because $\delta\delta' \neq 1$. If $d(Z, Z') = 2$, then $\delta\delta' \in I^*$ and has centre at distance 2 from both Z and Z' , and this contradicts $(X_2, X_3, \dots, X_8) \subseteq \Delta(\langle \delta'\delta \rangle)$. Suppose then that Z and Z' are opposites. Choose $Y \in \Delta_4(Z) \cap \Delta_4(Z')$ with $Y \neq X_5$. Then $d(X_2, Y) = 5$ or 7 because Y and X_5 are opposites. We argue that $d(X_3, Y) = 6$. Certainly this is true if $d(X_2, Y) = 5$. Suppose that $d(X_2, Y) = 7$. Then the vertex Y' in $\langle X_2, Y \rangle$ which is adjacent to X_2 is X_3 because $d(Y', Z) \leq 6$ by Lemma 1.5(i). Now $[\alpha_3, \delta\delta'] = 1$ because $\delta\delta'$ fixes X_3 . Hence Y^{α_3} is fixed by $\delta\delta'$. But $d(X_3, Y) = 6$ implies that Y^{α_3} is opposite to Z , and Lemma 1.5(ii) is contradicted. Therefore $d(Z, Z') = 6$ or 4 as claimed.

If $d(Z, Z') = 6$, then $d(Z, Z^{\delta'}) = 4$, and $\delta\delta^{\delta'}$ also fixes each vertex in (X_2, X_3, \dots, X_8) . So we may assume that $d(Z, Z') = 4$. Then $\delta\delta'$ is an involution by Lemma 2.1. Now $\delta\delta'$ fixes X_8 and X_7 , but $\delta\delta'$ fixes no vertex in $\Xi_1^*(X_8) - \{X_7\}$ by Lemma 1.5(i). Therefore $[M(X_8), \delta\delta'] \leq M(X_8)$ and $C_{M(X_8)}(\langle \delta\delta' \rangle) = G(X_7)$ by Lemma 3.10. It follows that $M(X_8) = G(X_7) \times G(X_9) \cong Z_2 \times Z_2$.

COMPLETION OF THE PROOF OF THEOREM 3.9. Set $S = \langle M(X_0), M(X_{10}) \rangle$, and let Ω be the orbit of X_0 under S . Then $X_{10} \in \Omega$ and $\Omega \subseteq \Delta_3(X_{13})$. Then $d(Y, Y') = 6$ for all distinct $Y, Y' \in \Omega$. For suppose $Y \in \Omega$ is such that $d(X_0, Y) \neq 6$. Then X_{15} belongs to $\langle X, X_1 \rangle$ for each $X \in \mathfrak{J}^*$ which is adjacent to Y . Thus $X_{15} \in \mathfrak{J}^*$ by Corollary 3.8 and we contradict our choice of Σ . Now Lemma 3.12 and Hering's trivial normalizer intersection theorem [5] imply that $S \cong L_2(4)$, and (Ω, S) is the natural 2-transitive group space. In particular, $\langle \alpha_1, \alpha_9 \rangle \cong D_6$ or D_{10} , where α_1 and α_9 are as in Lemma 3.12. We consider these two possibilities in turn, and show that they both lead to a contradiction. This shows that our non-convex component does not exist, and completes the proof of Theorem 3.9.

First, suppose that $\langle \alpha_1, \alpha_9 \rangle \cong D_6$. Then $\alpha_1\alpha_9$ fixes two distinct vertices $Y, Y' \in \Omega - \{X_0, X_{10}\}$. Let Y_1 be the vertex in $\langle Y, X_{13} \rangle$ which is adjacent to Y , and Y'_1 the vertex in $\langle Y', X_{13} \rangle$ which is adjacent to Y' . Then $d(Y_1, Y'_1) = 4$ and X_{13} belongs to $\langle Y_1, Y'_1 \rangle$. Certainly both Y_1 and Y'_1 are fixed by $\alpha_1\alpha_9$, so $d(Y_1, X_5) = 6 = d(Y'_1, X_5)$ by Lemma 1.5(ii). Set $\langle Y_1, X_5 \rangle = (Y_1, Y_2, \dots, Y_7 = X_5)$ and $\langle Y'_1, X_5 \rangle = (Y'_1, Y'_2, \dots, Y'_7 = X_5)$. Then $\langle Y_1, Y'_1 \rangle \cup \langle Y_1, X_5 \rangle \cup \langle Y'_1, X_5 \rangle$ is an apartment in $\Delta(\langle \alpha_1\alpha_9 \rangle)$. If $Y = Y_2$, then $d(Y, Y'_5) = 7$,

and there exists a vertex in $\langle Y, Y'_5 \rangle$ which is opposite to both X_9 and X_{13} . This contracts Lemma 1.5(ii) because such a vertex is fixed by $\alpha_1\alpha_9$. Therefore $Y = Y_2$. But now $d(X, X_5) \leq 6$ for each $X \in \Xi_1^*(Y)$. Hence Corollary 3.8 implies that $Y_3 \in \mathfrak{J}^*$. This means that $\alpha_1\alpha_9$ centralizes $G(Y_3)$, which is impossible because $G(Y_3) < S$ and $C_S(\langle \alpha_1\alpha_9 \rangle) = \langle \alpha_1\alpha_9 \rangle$. Therefore $\langle \alpha_1, \alpha_9 \rangle \cong D_6$.

Finally, we consider the possibility $\langle \alpha_1, \alpha_9 \rangle \cong D_{10}$. In this case, $\langle \alpha_1\alpha_9 \rangle$ is sharply transitive on Ω . In particular $\Omega \subseteq \Delta_5(X_5)$. Choose $X \in \Xi_1^*(X_0) - \{X_1\}$ and $1 \neq \alpha \in G(X)$. Then $\Omega = \Omega^\alpha \subseteq \Delta_5(X_5^\alpha)$. In particular, $d(X_{10}, X_5^\alpha) = 5$. This is absurd because X_4^α is opposite to X_{10} and adjacent to X_5^α . Therefore $\langle \alpha_1, \alpha_9 \rangle \cong D_{10}$.

4. CONNECTED COMPONENTS OF TYPE (E)

Theorem 3.9 tells us that the connected components of Ξ are convex and of one of three possible types. In this section, we shall classify the pairs (Ξ^*, E^*) when Ξ^* is a connected component of type (E_4) or type (E_8) . We begin with the first type. The notation is as explained at the outset of Section 3.

THEOREM 4.1. *Suppose that Ξ^* is a connected component of type (E_4) . Then Ξ^* is the doubled flag complex of a Moufang generalized quadrangle associated with a group of type Sp_4 defined over a finite field \mathbb{F}_q of characteristic 2, and $\tilde{E}^* \cong Sp_4(q)$. Furthermore, $K^* = C_{G^*}(E^*)$, and either E^* is a perfect central extension of \tilde{E}^* , or $q = 2$.*

PROOF. By Proposition 3.7, $G(X)$ is an elementary abelian 2-group for each $X \in \mathfrak{J}^*$. Therefore the theorem follows from the main result in [2, p. 1]. Alternatively, if $|G(X)| > 2$ for each $X \in \mathfrak{J}^*$, then the result follows from [15, Theorem 1.1], whilst the case $G(X) \cong \mathbb{Z}_2$ for some $X \in \mathfrak{J}^*$ may be handled separately.

The conclusion for components of type (E_8) is analogous, but somewhat more difficult to prove.

THEOREM 4.2. *Suppose that Ξ^* is a connected component of type (E_8) . Then Ξ^* is the flag complex of a Moufang generalized octagon associated with a Ree group of type 2F_4 defined over some finite field \mathbb{F}_q (here $q = 2^n$ with n odd). Furthermore, either $\tilde{E}^* \cong {}^2F_4(q)$, or $q = 2$ and $\tilde{E}^* \cong {}^2F_4(2)'$. In each case, $K^* = C_{G^*}(E^*)$ and E^* is a perfect central extension of \tilde{E}^* .*

PROOF. Throughout, we let $s+1$ be the common valency in Ξ^* of the vertices in \mathfrak{J}^* and $t+1$ the common valency in Ξ^* of the vertices in \mathfrak{J}^* . Thus $s, t \geq 2$. Moreover, $2st$ is a perfect square by a theorem of Feit and Higman [3].

By Proposition 3.7, $G(X)$ is an elementary abelian 2-group of order independent of the choice of $X \in \mathfrak{J}^*$. We set $q = |G(X)|$.

Let $\Sigma = (X_0, X_1, \dots, X_{15})$ be an apartment in Ξ^* with $X_1 \in \mathfrak{J}^*$. For each $i \in \mathbb{Z}/16\mathbb{Z}$ define:

$$\begin{aligned} H_{2i+1} &= \langle G(X) | X \in \Xi_4^*(X_{2i+1}) \cap \Xi_4^*(X_{2i+9}) \rangle, \\ U_{2i} &= \langle u \in E^* | \Delta_{[2]}(X_{2i-1}) \cup \Delta_{[2]}(X_{2i+1}) \subseteq \Delta(\langle u \rangle) \rangle, \\ V_{2i+1} &= \langle U_{2i}, U_{2i+2} \rangle. \end{aligned}$$

Of course, H_{2i+1} depends only upon X_{2i+1} and X_{2i+9} , U_{2i} depends only upon X_{2i-1} and X_{2i+1} , and V_{2i+1} depends only upon X_{2i-1} and X_{2i+3} .

The first stage of the proof is to determine s and t and the structure of the groups $M_{2i} = M(X_{2i})$ and H_{2i+1} (cf. Lemmas 4.10 and 4.11). We then study the groups U_{2i} and V_{2i+1} and determine the root groups of Ξ^* which have centre in \mathfrak{F} (cf. Lemmas 4.20 and 4.21). In the final stage of the proof we handle the cases $q \neq 2$ and $q = 2$ separately. In the former case we show that all root groups of Ξ^* in \tilde{E}^* are transitive, and then use Tits's [14] classification to identify Ξ^* . When $q = 2$, we identify \tilde{E}^* (and hence Ξ^*) by applying Parrott's [10] characterization of ${}^2F_4(2)'$.

LEMMA 4.3. *If $q = 2$, then $H_{2i+1} = O(H_{2i+1}) \rtimes G(X_{2i-3})$, whilst if $q \neq 2$, then $H_{2i+1} \cong L_2(q)$, $Sz(q)$, $U_3(q)$ or $SU(3, q)$ and $(\Xi_1^*(X_{2i+1}), H_{2i+1})$ is the natural 2-transitive group space.*

PROOF. Use [5] and argue as in the proof of Lemma 2.3.

COROLLARY 4.4. *If $q \neq 2$, then $s = q$, q^2 or q^3 .*

LEMMA 4.5. *Assume that $q \neq 2$, and let Z be the vertex stabilizer of Σ in $\langle G(X_{2i-3}), G(X_{2i+5}) \rangle$. Then $Z \cong Z_{q-1}$. Moreover, Z is semi-regular on $\Xi_1^*(X_{2i-2}) - \{X_{2i-3}, X_{2i-1}\}$ and regular (by conjugation) on $G(X_{2i-1}) - \{1\}$.*

PROOF. Set $S = \langle G(X_{2i-3}), G(X_{2i+5}) \rangle$. Then $q \neq 2$ and Lemma 4.3 imply that $S \cong L_2(q)$ or $Sz(q)$ and $Z = N_S(G(X_{2i-3})) \cap N_S(G(X_{2i+5}))$. Therefore $Z \cong Z_{q-1}$. Let $D = N_S(Z)$. Thus $D \cong D_{2(q-1)}$, and each involution in D belongs to I^* and has centre in the set $\Xi_4^*(X_{2i+1}) \cap \Xi_4^*(X_{2i+9}) - \{X_{2i-3}, X_{2i+5}\}$.

Let $z \in Z$ with $z \neq 1$, and let $X \in \Xi_1^*(X_{2i-2}) - \{X_{2i-3}, X_{2i-1}\}$. Since we can write $z = \delta\gamma$ with $\delta, \gamma \in D - Z$, it follows from Lemma 1.5(ii) that $X^z \neq X$. Similarly, if $1 \neq \alpha \in G(X_{2i-1})$, then z does not fix X_{2i+5} . Therefore $[\alpha, z] \neq 1$.

COROLLARY 4.6. $t - 1 \equiv 0 \pmod{q-1}$.

LEMMA 4.7. *The stabilizer $E_{X_j}^*$ is 2-transitive on $\Xi_1^*(X_j)$.*

PROOF. We may assume that $j = 2$ or 3 . Let $X \in \Xi_1^*(X_2) - \{X_1, X_3\}$ and $Y \in \Xi_1^*(X_3) - \{X_2, X_4\}$. Choose $X' \in \Xi_1^*(Y)$, with $X' \neq X_3$. Clearly, it suffices to find $\delta, \delta' \in E_{X_5}^*$ with $X_1^\delta = X$ and $X_1^{\delta'} = X'$.

Choose $\alpha \in G(X_1)$ and $\beta \in G(X_9)$ with $\alpha \neq 1 \neq \beta$. Then $\langle \alpha, \beta \rangle \cong D_{2m}$ with m odd by Lemma 4.3. Therefore there exists $\gamma \in \langle \alpha, \beta \rangle \cap I^*$ which interchanges X_1 and X_9 , and fixes X_5 . Apply the same argument with X (resp. X') in place of X_1 to produce a ρ (resp. ρ') which interchanges X (resp. X') and X_9 as well as fixing X_5 . Now set $\delta = \gamma\rho$ and $\delta' = \gamma\rho'$.

LEMMA 4.8. *Assume that there exist $\alpha \in G(X_{2i-1})$ and $\beta \in G(X_{2i+1})$ with $\alpha \neq 1 \neq \beta$ and $\alpha\beta \in I^*$. Then $\{G(X) | X \in \Xi_1^*(X_{2i})\}$ is a partition of the elementary abelian 2-group $M(X_{2i})$.*

PROOF. Obviously each element in $M(X_{2i}) \cap I^*$ has centre in $\Xi_1^*(X_{2i})$. In view of this and Lemma 4.7, it is sufficient to prove that $G(X_{2i-1}) \times G(X_{2i+1}) \subseteq I^* \cup \{1\}$. In particular, our hypothesis means that there is nothing more to show when $q = 2$. Assume that $q \neq 2$, and let Z be as in Lemma 4.5. Then Z centralizes $G(X_{2i+1})$ and Z is regular on $G(X_{2i-1}) - \{1\}$ by Lemma 4.5. Therefore $G(X_{2i-1})\beta = \alpha^Z\beta \cup \{\beta\} = (\alpha\beta)^Z \cup \{\beta\} \subseteq I^*$.

Now $G(X_{2i-1}) \times G(X_{2i+1}) \subseteq I^* \cup \{1\}$ follows upon interchanging the labellings of X_{2i-1} and X_{2i+1} .

LEMMA 4.9. *The stabilizer of X_{2i+4} in $M(X_{2i})$ is precisely $G(X_{2i+1})$.*

PROOF. We may assume that $i = 0$. Certainly $G(X_1) \leq M(X_0)_{X_4}$. Let $x \in M(X_0)$ fix X_4 , and choose $\alpha \in G(X_5)$ with $\alpha \neq 1$. Then $[x, \alpha] \in M(X_4)$, and as such is an involution. We have $\Delta_{[2]}(X_1) \subseteq \Delta(\langle [x, \alpha] \rangle)$ because $x \in M(X_0)$ and α fixes X_1 , and $\Delta_{[3]}(X_4) \subseteq \Delta(\langle [x, \alpha] \rangle)$. Hence, $[x, \alpha] \in G(X_3)$ by Lemma 2.5. If $[x, \alpha] \neq 1$, then Lemma 4.8 (applied in an appropriate apartment) and the transitivity of E^* on \mathfrak{F}^* imply that $M(X_0)$ is partitioned by $\{G(X)|X \in \Xi_1^*(X_0)\}$. But then $x \in G(X_1)$ is obvious. Conversely, if $[x, \alpha] = 1$, then $x \in G(X_1)$ follows from $x^2 = 1$ and Lemma 2.5.

LEMMA 4.10. *The elementary abelian 2-group $M(X_{2i}) = G(X_{2i-1}) \times G(X_{2i+1})$ and is partitioned by $\{G(X)|X \in \Xi_1^*(X_{2i})\}$. Moreover, $s = q^2$ and $t = q$.*

PROOF. We may assume that $i = 1$. The cases $q = 2$ and $q \neq 2$ are treated separately.

Case I. Assume that $q = 2$.

Suppose that we have proved the first statement in the lemma. Then clearly $t = 2$, and, as $2st$ is a perfect square, it follows that s is a perfect square. However, $s \leq t^2$ by a result of Higman [7, Theorem 3.2], and so we force $s = 4$. Choose $X \in \Xi_1^*(X_2) - \{X_1, X_3\}$. The argument used in the proof of Lemma 4.7 shows that we can find γ (resp. γ') in $I^* \cap E_{X_s}^*$, which interchanges X_1 (resp. X) and X_9 . Therefore $\gamma\gamma' \neq 1$, and $\gamma\gamma'$ fixes each vertex in $\langle X_2, X_5 \rangle$.

Let Z (resp. Z') be the centre of γ (resp. γ'). Then Z and Z' are at distance 4 from X_5 , and at distance 7 from X_2 . We argue that $d(Z, Z') = 4$ or 6. Certainly, $Z \neq Z'$ because $\gamma\gamma' \neq 1$, and $d(Z, Z') \neq 2$ because otherwise $\gamma\gamma'$ would not be able to fix X_2 by Lemma 1.5(i). It remains to exclude the possibility that Z and Z' are opposites.

By way of a contradiction, assume that $d(Z, Z') = 8$. Choose $W \in \mathfrak{F}^*$ so that $d(Z, W) = 4 = d(Z', W)$ but $W \neq X_5$. Then Lemma 1.5(ii) shows that $d(W, X_3) = 6$. Set $G(X_3) = \langle \alpha_3 \rangle$. Then $[\alpha_3, \gamma\gamma'] = 1$ because $\gamma\gamma'$ fixes X_3 . Thus α_3 leaves $\Delta(\langle \gamma\gamma' \rangle)$ invariant. In particular, W^{α_3} is fixed by $\gamma\gamma'$. However, W^{α_3} is opposite to X_5 and to Z , and this contradicts Lemma 1.5(ii).

If $d(Z, Z') = 6$, then $d(Z, Z') = 4 = d(X_5, Z')$, and $\gamma\gamma^{\gamma'}$ also fixes each vertex in $\langle X_2, X_5 \rangle$. After replacing γ' by $\gamma^{\gamma'}$, we may assume that $d(Z, Z') = 4$. Therefore $\gamma\gamma'$ is an involution by Lemma 2.1. Since $\gamma\gamma'$ fixes X_2 , it normalizes $M(X_2)$. Moreover, $\gamma\gamma'$ fixes no vertex in $\Xi_1^*(X_2) - \{X_3\}$ because it can fix no vertex which is opposite Z by Lemma 1.5(i).

Now let $X' \in \Xi_1^*(X_2) - \{X_3\}$, and set $G(X') = \langle \alpha \rangle$. Then $[\alpha, \gamma\gamma'] \in M(X_2)$ and $[\alpha, \gamma\gamma'] \neq 1$. As $[\alpha, \gamma\gamma']$ centralizes $\gamma\gamma'$, it leaves $\Delta(\langle \gamma\gamma' \rangle)$ invariant. In particular, $Z^{[\alpha, \gamma\gamma']}$ is fixed by $\gamma\gamma'$. Since $Z^{[\alpha, \gamma\gamma']}$ cannot be opposite Z , it follows that $[\alpha, \gamma\gamma']$ fixes a vertex in $\Xi_1^*(X_5) - \{X_4\}$. Therefore, $[\alpha, \gamma\gamma'] \in G(X_3)$ by Lemma 4.9. But now $\{G(X')|X' \in \Xi_1^*(X_2)\}$ is a partition of $M(X_2)$ by Lemma 4.8. Since $[G(X'), \gamma\gamma'] \leq G(X_3)$ for all $X' \in \Xi_1^*(X_2)$, it follows that $M(X_2) = G(X_1) \times G(X_3)$.

Case II. Assume that $q \neq 2$.

Suppose we have shown that $\{G(X)|X \in \Xi_1^*(X_2)\}$ is a partition of $M(X_2)$. Then Corollary 4.4 and Lemma 4.9 imply that $M(X_2) = q^n$ with $n \leq 4$. Since $t+1 = (q^n - 1)/(q - 1)$, it follows from Corollary 4.6 that $t = q$ and $M(X_2) = G(X_1) \times G(X_3)$ (i.e. $n = 2$). Now $2st$, a perfect square, and Corollary 4.4 force $s = q^2$ (with q an odd power of 2).

It remains to prove that $\{G(X)|X \in \Xi_1^*(X_2)\}$ really does partition $M(X_2)$. Assume that this is false. Then $t(q^2 - q) + q \leq |M(X_2)|$ by Lemma 4.8. Since $|M(X_2)| \leq sq$ by Lemma 4.9, we conclude that

$$t(q - 1) \leq s - 1. \quad (1)$$

Since $t \geq 2$, it now follows from Lemma 4.3 that either $s = q^2$ and $H_5 \cong Sz(q)$, or $s = q^3$ and $H_5 \cong U_3(q)$ or $SU(3, q)$. We argue that

$$s = q^3 \text{ and } t - 1 \equiv 0 \pmod{4}. \quad (2)$$

Clearly, this follows from the fact that $2st$ is a perfect square once it is known that t is odd.

Let P be the Sylow 2-subgroup of H_5 which contains $G(X_1)$. Then P fixes each vertex in $\langle X_1, X_5 \rangle$, and is sharply transitive on $\Xi_1^*(X_5) - \{X_4\}$ by Lemma 4.3. We argue that P fixes no vertex in $\Xi_1^*(X_2) - \{X_1, X_3\}$. Then $t - 1 \equiv 0 \pmod{2}$ is obvious. By way of a contradiction, assume that $X \in \Xi_1^*(X_2) - \{X_1, X_3\}$ is fixed by P . Choose $\alpha \in G(X)$, with $\alpha \neq 1$, and select $u \in P$ such that $X_6^{\alpha u} = X_6$. Then $[\alpha, u^{-1}] \in G(X)$ because u fixes X , and $u^2 \in G(X_1)$ because $P^2 = G(X_1)$. But $[\alpha, u^{-1}]u^2 = (\alpha u)^2$ fixes X_6 , so $[\alpha, u^{-1}]u^2 \in G(X_3)$ by Lemma 4.9. Therefore Lemma 4.8 and our assumption on $M(X_2)$ imply $[\alpha, u^{-1}] = 1 = u^2$. Therefore $u \in G(X_1)$ because $\Omega(P) = G(X_1)$. Thus $\alpha u \in G(X_3)$ by Lemma 4.9, and hence Lemma 4.8 contradicts our assumption on $M(X_2)$.

From (2), we have $H_5 \cong U_3(q)$ or $SU(3, q)$. Set $C = C_{H_5}(\langle G(X_1), G(X_9) \rangle)$. So C is cyclic and has order $(q + 1)/(3, q + 1)$ or $q + 1$, according to whether $H_5 \cong U_3(q)$ or $H_5 \cong SU(3, q)$. Suppose that $X^a = X$ for some $X \in \Xi_1^*(X_2) - \{X_1, X_3\}$ and some $a \in C$. Then a centralizes some $\alpha \in G(X) - \{1\}$. Suppose that $a \notin Z(H_5)$. Then the fixed set of a in $\Xi_1^*(X_5)$ is precisely $X_6^{G(X_1)} \cup \{X_4\}$. Hence there exists $\alpha' \in G(X_1)$, with $X_6^\alpha = X_6^{\alpha'}$. Therefore $\alpha\alpha' \in G(X_3)$ by Lemma 4.9, and once again Lemma 4.8 contradicts the hypothesis on $M(X_2)$. Therefore $a \in Z(H_5)$, and we conclude that

$$t - 1 \equiv 0 \pmod{(q + 1)/(3, q + 1)}. \quad (3)$$

From (1), (2), (3) and Corollary 4.6, we deduce that

$$4(q^2 - 1)/(3, q + 1) \leq q^2 + q.$$

This means $q = 2$, which contradicts the hypothesis.

This enables us to improve Lemma 4.3.

LEMMA 4.11. *If $q = 2$, then $H_{2i+1} \cong D_{10}$, whilst if $q \neq 2$, then $H_{2i+1} \cong Sz(q)$ and $(\Xi_1^*(X_{2i+1}), H_{2i+1})$ is the natural 2-transitive group space.*

PROOF. If $q \neq 2$, then the lemma follows directly from Lemma 4.3 and the fact that $s = q^2$ by Lemma 4.10. Suppose that $q = 2$. Then $\langle G(X_{2i-3}), G(X_{2i+5}) \rangle \cong D_{2m}$ with m odd by Lemma 4.3. Therefore $m = 3$ or 5 because $s = 4$ by Lemma 4.10. If $m = 3$, then $\langle G(X_{2i-3}), G(X_{2i+5}) \rangle$ is centralized by some involution in I^* which has its centre in $\Xi_4^*(X_{2i+1}) \cup \Xi_4^*(X_{2i+9})$. But now consideration of the fixed structure of the cyclic stem of $\langle G(X_{2i-3}), G(X_{2i+5}) \rangle$ leads to a contradiction. Therefore $m = 5$, and hence $\langle G(X_{2i-3}), G(X_{2i+5}) \rangle$ is transitive on $\Xi_4^*(X_{2i+1}) \cup \Xi_4^*(X_{2i+9})$. Consequently $H_{2i+1} = \langle G(X_{2i-3}), G(X_{2i+5}) \rangle$.

Thus we have determined the parameters s and t of Ξ_4^* and the structure of the groups $M_{2i} = M(X_{2i})$ and H_{2i+1} . We now turn to the groups U_{2i} and V_{2i+1} .

LEMMA 4.12. *For each i , we have U_{2i} is an elementary abelian 2-group.*

PROOF. Let $u \in U_{2i}$, and choose $\alpha \in G(X_{2i+5})$ with $\alpha \neq 1$. Then $[u, u^\alpha] \in U_{2i}$ because u^α fixes both X_{2i-1} and X_{2i+1} , and $[\alpha, \alpha^u] = 1$ because u fixes X_{2i+3} and Lemma 2.1 may be applied. Therefore $\alpha^{u^2} \alpha = [u, u^\alpha] \in U_{2i}$. In particular, $\alpha^{u^2} \alpha$ fixes X_{2i-3} . However, X_{2i-3} is opposite X_{2i+5} , so $\alpha^{u^2} \alpha = 1$ by Lemma 1.5(i). Therefore u^2 fixes X_{2i+5} . Similarly, u^2 fixes X_{2i-5} . Hence u^2 fixes each vertex in Σ , and so $u^2 = 1$ by Tits's Lemma 1.2.

COROLLARY 4.13. *The intersection $U_{2i} \cap U_{2i+2} = G(X_{2i+1})$ and $q^2 \leq |U_{2i}| \leq q^4$.*

PROOF. From Lemmas 4.11 and 2.5 we conclude that $(U_{2i})_{X_{2i+5}} = G(X_{2i+1})$. The bounds on $|U_{2i}|$ now follow from Lemma 4.10.

LEMMA 4.14. *For each i , we have V_{2i+1} is a 2-group and $\Phi(V_{2i+1}) < G(X_{2i+1})$.*

PROOF. This follows from Lemma 4.12 and Corollary 4.13 because $[U_{2i}, U_{2i+2}] \leq U_{2i} \cap U_{2i+2}$.

In the following, set $\bar{V}_{2i+1} = V_{2i+1}/G(X_{2i+1})$. Then \bar{V}_{2i+1} is an elementary abelian 2-group, which will be written additively and viewed as a vector space over the field \mathbb{F}_2 .

LEMMA 4.15. *The vector space \bar{V}_{2i+1} is a faithful $\mathbb{F}_2 H_{2i+1}$ -module, and for $\alpha \in G(X_{2i-3})$ with $\alpha \neq 1$ we have*

$$[\bar{V}_{2i+1}, \alpha] = \bar{U}_{2i} = C_{\bar{V}_{2i+1}}(\alpha).$$

(That is, α is a shear on \bar{V}_{2i+1} with axis \bar{U}_{2i} .)

PROOF. Since $G(X_{2i-3}) \leq U_{2i-2}$, we have $[U_{2i}, G(X_{2i-3})] \leq U_{2i} \cap U_{2i-2} = G(X_{2i-1})$ by Corollary 4.13. On the other hand, $[U_{2i}, G(X_{2i-3})] \leq G(X_{2i-3})$ because U_{2i} fixes X_{2i-3} . Therefore

$$[U_{2i}, G(X_{2i-3})] = \{1\}.$$

It is straightforward to see that $[U_{2i+2}, G(X_{2i-3})] \leq U_{2i}$, so \bar{V}_{2i+1} is an $\mathbb{F}_2 G(X_{2i-3})$ -module. Similarly, \bar{V}_{2i+1} is an $\mathbb{F}_2 G(X_{2i+5})$ -module. From Lemma 4.11 we see that $H_{2i+1} = \langle G(X_{2i-3}), G(X_{2i+5}) \rangle$. Therefore \bar{V}_{2i+1} is an $\mathbb{F}_2 H_{2i+1}$ -module.

If $\alpha \in G(X_{2i-3})$, and if $\alpha \neq 1$, then

$$U_{2i+2} \cap U_{2i+2}^\alpha = G(X_{2i+1}),$$

where this follows from Corollary 4.13 applied in any apartment containing $\langle X_{2i+2}, X_{2i+2}^\alpha \rangle$. Therefore

$$[\bar{V}_{2i+1}, \alpha] \leq \bar{U}_{2i} = C_{\bar{V}_{2i+1}}(\alpha)$$

because $\bar{V}_{2i+1} = \bar{U}_{2i} \oplus \bar{U}_{2i+2}$. Similarly,

$$\bar{U}_{2i+2} = C_{\bar{V}_{2i+1}}(\beta) \quad \text{for all } 1 \neq \beta \in G(X_{2i+5}).$$

But now $\dim_{\mathbb{F}_2} \bar{U}_{2i} = \dim_{\mathbb{F}_2} \bar{U}_{2i+2}$ because $G(X_{2i-3})$ and $G(X_{2i+5})$ are conjugate in H_{2i+1} . Therefore the inequality above is actually an equality.

LEMMA 4.16. *As an $\mathbb{F}_2 H_{2i+1}$ -module, \bar{V}_{2i+1} is one of the following types:*

- (i) $q = 2$ and \bar{V}_{2i+1} is isomorphic to the $\mathbb{F}_2 D_{10}$ -module obtained from the standard module for $L_2(4)$;
- (ii) $q \neq 2$ and \bar{V}_{2i+1} is isomorphic to the standard module for $Sz(q)$.

PROOF. The isomorphism types for H_{2i+1} are given in Lemma 4.11. Set

$$\mathfrak{R} = \{C_{\bar{V}_{2i+1}}(S) | S \in \text{Syl}_2(H_{2i+1})\}.$$

Then, from Lemma 4.15 and the structure of H_{2i+1} , it follows that \mathfrak{R} is a partial spread of \bar{V}_{2i+1} , and the involutions in H_{2i+1} are all \mathfrak{R} -elations (i.e. $[\bar{V}_{2i+1}, \alpha] = C_{\bar{V}_{2i+1}}(\alpha) \in \mathfrak{R}$, for every involution $\alpha \in H_{2i+1}$). Let $x \in H_{2i+1}$ have order 5. Then $x = \alpha\beta$, where α and β are involutions in H_{2i+1} , and $C_{\bar{V}_{2i+1}}(\alpha) \neq C_{\bar{V}_{2i+1}}(\beta)$. An easy argument (cf. [6, 2.13]) shows that $C_{\bar{V}_{2i+1}}(x) = (0)$. If $q = 2$, it follows that $|\bar{V}_{2i+1}| = 2^m$ with $m \equiv 0 \pmod{4}$. If $q \neq 2$ (so $H_{2i+1} \cong \text{Sz}(q)$), then \bar{V}_{2i+1} is a direct sum of copies of the standard module for $\text{Sz}(q)$ by a result due to Martineau [9]. So in any case, $|\bar{V}_{2i+1}| = q^m$ with $m \equiv 0 \pmod{4}$. From Corollary 4.13, we see that $q^2 \leq |\bar{V}_{2i+1}| \leq q^6$. Hence $m = 4$. But now (ii) is clear, and (i) follows because \mathfrak{R} is a spread.

COROLLARY 4.17. *We can write $U_{2i} = M(X_{2i}) [G(X_{2i-3}), G(X_{2i+3})]$.*

PROOF. This follows because Lemma 4.16 tells us that U_{2i} has order q^3 .

LEMMA 4.18. *The centralizer in E^* of \bar{V}_{2i+1} fixes every vertex in $\Xi_1^*(X_{2i+1})$.*

PROOF. Let $x \in C_{E^*}(\bar{V}_{2i+1})$. Then x fixes X_{2i} because otherwise Corollary 4.13 applied in an apartment containing X_{2i} and X_{2i}^x is contradicted. Since H_{2i+1} is transitive on $\Xi_1^*(X_{2i+1})$ by Lemma 4.11, and as \bar{V}_{2i+1} is an $\mathbb{F}_2 H_{2i+1}$ -module by Lemma 4.15, repeated use of the above argument shows that x fixes each vertex in $\Xi_1^*(X_{2i+1})$.

LEMMA 4.19. *The 2-group $U_{2i} = C_{U_{2i}}(\bar{V}_{2i+3}) \times G(X_{2i-1})$.*

PROOF. Let \mathfrak{R} be the orbit of \bar{U}_{2i+4} under H_{2i+3} . Then \mathfrak{R} is a spread of \bar{V}_{2i+3} by Lemma 4.16. Let $u \in U_{2i}$. Then Corollary 4.13 (applied in appropriate apartments) implies that $\bar{U}_{2i+4}^u \cap \bar{U}_{2i+4}^h = (0)$ for all $h \in H_{2i+3}$ with $X_{2i+4}^h \neq X_{2i+4}^u$. This shows that u leaves the spread \mathfrak{R} invariant. Therefore if $u \notin C_{U_{2i}}(\bar{V}_{2i+3})$, then $[\bar{V}_{2i+3}, u] = \bar{U}_{2i+2} = C_{\bar{V}_{2i+3}}(u)$ (i.e. U_{2i} acts on \bar{V}_{2i+3} as a group of \mathfrak{R} -elations with axis \bar{U}_{2i+2}). If $q \neq 2$, then $U_{2i}/C_{U_{2i}}(\bar{V}_{2i+3}) \cong G(X_{2i-1})$ follows from the structure of \mathfrak{R} . If $q = 2$, then $U_{2i}/C_{U_{2i}}(\bar{V}_{2i+3}) \not\cong G(X_{2i-1})$ implies that $\langle H_{2i-3}, U_{2i} \rangle$ induces $L_2(4)$ on \bar{V}_{2i+1} . From this it follows that there exists $u \in U_{2i}$ such that if $G(X_{2i+7}) = \langle \alpha \rangle$, then $\langle \alpha, \alpha^u \rangle \cong D_6$, which contradicts Lemma 4.11 (applied in a suitable apartment). But now Lemma 4.19 is clear.

LEMMA 4.20. *Define $R_{2i} = C_{U_{2i}}(\bar{V}_{2i+3}) \cap C_{U_{2i}}(\bar{V}_{2i-3})$. Then R_{2i} is sharply transitive on $\Xi_1^*(X_{2i+4}) - \{X_{2i+3}\}$, and $[M(X_{2i+4}), R_{2i}] = G(X_{2i+3})$.*

PROOF. Let $\Psi = (X_{2i-4}, X_{2i-3}, \dots, X_{2i+4})$. Then R_{2i} fixes each vertex in Ψ and each vertex in Δ which is adjacent to X_{2i} or X_{2i+1} . Therefore R_{2i} is semi-regular on $\Xi_1^*(X_{2i+4}) - \{X_{2i+3}\}$. Hence $|R_{2i}| \leq q$ because X_{2i+4} has valency $q+1$ in Ξ^* by Lemma 4.10. On the other hand, Lemma 4.19 and U_{2i} elementary abelian of order q^3 imply that $|R_{2i}| \geq q$. This proves the first part of the lemma.

Since R_{2i} fixes X_{2i+4} , it normalizes $M(X_{2i+4})$, and $[M(X_{2i+4}), R_{2i}] \leq [U_{2i+4}, U_{2i}] \leq U_{2i+2}$. Therefore $[M(X_{2i+4}), R_{2i}] \leq M(X_{2i+4}) \cap U_{2i+2} \leq U_{2i+4} \cap U_{2i+2} \leq G(X_{2i+3})$ by Corollary 4.13. But now the second part of the lemma follows from the first.

COMPLETION OF THE PROOF OF THEOREM 4.2. From Lemma 4.20 and the transitivity of E^* on \mathfrak{R}^* , it follows that E^* is perfect. A straightforward argument (cf. [16, 4.1]) shows that $K^* = C_{G^*}(E^*)$. Therefore E^* is a perfect central extension of \tilde{E}^* . Clearly, from now on, we can assume that $\Xi^* = \Delta$.

LEMMA 4.21. *Let Ψ be a root in Δ having centre in \mathfrak{J} . Then $E(\Psi)$ is transitive on the apartments of Δ which contain Ψ .*

PROOF. Write $\Psi = (X_0, X_1, \dots, X_8)$, and let $\Sigma = (X_0, X_1, \dots, X_{15})$ be an apartment containing Ψ . From Lemma 4.18, the definition of U_4 and the definition of R_4 (given in Lemma 4.20), it follows that $R_4 \leq E(\Psi)$. Hence $E(\Psi)$ is transitive on the apartments which contain Ψ by Lemma 4.20.

Case I. Assume that $q \neq 2$.

In this case, we argue that Δ satisfies the Moufang condition with E being generated by all root automorphisms of Δ . Theorem 4.2 then follows from Tits's classification [14].

In view of Lemma 4.21, we only have to consider roots with centres in \mathfrak{J} . So let $\Phi = (X_1, X_2, \dots, X_9)$ be such a root, and choose $\Sigma = (X_0, X_1, \dots, X_{15})$ to be an apartment in Δ which contains Φ . Let P be a Sylow 2-subgroup of H_1 which contains $G(X_5)$. By Lemma 4.11, it is sufficient to prove that $P \leq E(\Phi)$.

By Lemma 4.10, $M(X_4) = G(X_3) \times G(X_5)$, and $M(X_4)$ is partitioned by $\{G(X) \mid X \in \Delta_1(X_4)\}$. Moreover, $G(X_5) \leq Z(P)$, and P normalizes $G(X_3)$ because it fixes X_3 . But P leave the above partition invariant, hence we have $[M(X_4), P] = \{1\}$, and consequently, $\Delta_1(X_4) \subseteq \Delta(P)$. The same argument now shows that $\Delta_1(X_2) \subseteq \Delta(P)$.

By Lemma 4.11, $H_1 = \langle G(X_5), G(X_{13}) \rangle$ and $H_2 = \langle G(X_1), G(X_9) \rangle$. Therefore $[H_1, H_5] = \{1\}$ by Lemma 2.1. In particular, $[H_5, P] = \{1\}$. But H_5 is transitive on $\Delta_1(X_5)$, and X_9 lies in the orbit of X_1 under H_5 , again by Lemma 4.11. Consequently, it only remains to show that P fixes each vertex which is adjacent to X_3 .

Since P fixes $\langle X_1, X_3 \rangle$, we have $[U_2, P] \leq U_2$. The transitivity of H_5 on $\Delta_1(X_5)$ and $\Delta_1(X_5) \subseteq \Delta(P)$ imply that $\Delta_2(X_5) \subseteq \Delta(P)$. Therefore $[U_2, P] \leq U_4$. Hence $[U_2, P] \leq G(X_3)$ by Corollary 4.13. By Corollary 4.17, $U_4 = M(X_4)[G(X_1), G(X_7)]$. Therefore $[U_4, P] = \{1\}$ from what has already been shown. Hence $[V_3, P] \leq G(X_3)$ (i.e. P centralizes \bar{V}_3). Therefore $\Delta_1(X_3) \subseteq \Delta(P)$ by Lemma 4.18.

Case II. Assume that $q = 2$.

If $E(\Phi)$ is transitive on the apartments containing Φ for every root Φ with centre in \mathfrak{J} , then Δ is Moufang (by Lemma 4.21), and E is generated by all root automorphisms of Δ . Now, as with the case $q \neq 2$, Tits's theorem implies that $E \cong {}^2F_4(2)$. However, this is impossible because ${}^2F_4(2)$ is not perfect. Therefore we can find a root $\Phi = (X_1, X_2, \dots, X_9)$ in Δ with $X_5 \in \mathfrak{J}$ and $E(\Phi)$ not transitive on the apartments containing Φ . So $E(\Phi) = G(X_5)$ because $s = 4$ by Lemma 4.10. Furthermore, the arguments which were used in Case I to show that $P \leq E(\Phi)$ may be used to prove that any element in E which fixes each vertex in Φ belongs to $E(\Phi)$. Therefore, we have the following lemma.

LEMMA 4.22. *The stabilizer $E_{X_1, X_2, \dots, X_9} \leq E(\Phi) = G(X_5)$.*

Let $\Sigma = (X_0, X_1, \dots, X_{15})$ be an apartment in Δ which contains Φ . For each $i \in \mathbb{Z}/16\mathbb{Z}$ set $G(X_{2i+1}) = \langle \alpha_{2i+1} \rangle$, thus $\alpha_{2i+1}^2 = 1 \neq \alpha_{2i+1}$. Set $H = C_E(\alpha_1)$, so $H = E_{X_1}$. We shall determine the structure of H .

First, we consider H_{X_9} . Obviously $H_1 \triangleleft H_{X_9}$. However, $H_1 = \langle \alpha_5, \alpha_{13} \rangle \cong D_{10}$ by Lemma 4.11. Therefore $H_{X_9} = H_1$, by Lemma 4.22. Consequently, we have $|H| = 2 \cdot 5 \cdot |X_9^H|$. Since Δ has the parameters $s = 4$ and $t = 2$ (by Lemma 4.10), a trivial computation shows that $|\Delta_8(X_1)| = 2^{10}$. Obviously $X_9^H \subseteq \Delta_8(X_1)$, so we obtain the inequality $|H| \leq 2^{11} \cdot 5$. Now, let J be the kernel of the group space $(\Delta_1(X_1), H)$. Then H/J is a 2-transitive subgroup of S_5 by Lemma 4.7. Hence $2^2 \cdot 5 \cdot |J| \leq |H|$. Combining these two inequalities gives the following lemma.

LEMMA 4.23. $2^2 \cdot 5 \cdot |J| \leq |H| \leq 2^{11} \cdot 5$.

We consider the structure of J . By Lemma 1.2, J is semi-regular on $\Delta_8(X_1)$. Therefore J is a 2-group.

Let Ψ (resp. Ψ') be the root in Σ which has centre X_4 (resp. X_{14}). Then $E(\Psi) \cong Z_2 \cong E(\Psi')$ by Lemma 4.21 and the fact that $t = 2$. It follows that $[E(\Psi'), \langle \alpha_3 \rangle] = \langle \alpha_1 \rangle$ but $[E(\Psi'), E(\Psi)] \neq \langle \alpha_1 \rangle$.

Obviously, $E(\Psi), E(\Psi') < J$, so we have shown the following Lemma.

LEMMA 4.24. $\langle \alpha_1 \rangle < J'$, the inclusion being proper.

Obviously $Z(J)$ must fix both X_4 and X_{14} , so $t = 2$ implies $|Z(J) : Z(J)_{X_{13}, X_{14}, \dots, X_5}| \leq 2^2$. But the root $\Phi' = (X_{13}, X_{14}, \dots, X_5)$ is in the orbit of $\Phi = (X_1, X_2, \dots, X_9)$ under $\langle \alpha_3, \alpha_{11} \rangle$. Hence Lemma 4.22 implies that $Z(J)_{X_{13}, X_{14}, \dots, X_5} = \langle \alpha_1 \rangle$. This proves the following lemma.

LEMMA 4.25. The index $|Z(J) : \langle \alpha_1 \rangle| \leq 2^2$.

Let P be a Sylow 5-subgroup of H_1 . Then $P \cong Z_5$ and $\Delta(P) = \Delta_4(X_5) \cap \Delta_4(X_{13})$ because $H_1 = \langle \alpha_5, \alpha_{13} \rangle \cong D_{10}$. Since $C_J(P)$ necessarily leaves $\Delta(P)$ fixed, it follows that $C_J(P) \leq C_J(H_1)$. In particular, $C_J(P)$ fixes each vertex in the root $\Phi' = (X_{13}, X_{14}, \dots, X_5)$. Now the argument used above shows that $C_J(P) = \langle \alpha_1 \rangle$. Hence $P \cong Z_5$ and Lemma 4.25 imply the following lemma.

LEMMA 4.26. $C_J(P) = Z(J) = \langle \alpha_1 \rangle$.

Now Lemmas 4.26 and 4.24 and $P \cong Z_5$ imply that $|J' : Z(J)| \geq 2^4$. Moreover, Lemma 4.26, $P \cong Z_5$ and $J \neq Z(J)$ show that $|J : \Phi(J)| \geq 2^4$. Therefore $|J| \geq 2^9$, and J has nilpotency class at least 3. From Lemmas 4.26 and 4.23 and H/J a 2-transitive subgroup of S_5 , we now see that H has the following structure:

- (i) if $J = O_2(H)$, then J has order 2^9 and nilpotency class at least 3 (actually exactly 3);
- (ii) H/J is a Frobenius group of order 20;
- (iii) if P is Sylow 5-subgroup of H , then $C_J(P) = Z(J)$.

Since E is perfect, and as H is the centralizer in E of the involution of α_1 , (i)–(iii) and Parrot's characterization [10] imply that $E \cong {}^2F_4(2)'$. That Δ is the Moufang building associated to ${}^2F_4(2)$ now follows easily.

5. THE STRUCTURE OF Ξ

Theorem A is an immediate consequence of Theorems 3.9, 4.1 and 4.2. It remains to consider the case when Ξ is disconnected.

Given a connected component Ξ^* of Ξ , we say that Ξ^* is of type X if statement X in Theorem A holds.

Suppose that Ξ^* is a connected component of Ξ of type II j ($j = 1, 2, 3, 4$ or 5). Then there is a unique vertex Z in $\mathfrak{J}^* \cup \mathfrak{J}^*$ which satisfies

$$\Xi^* \subseteq \Xi_{[j-1]}(Z).$$

We call Z the *centre* of Ξ^* . Observe that if j is odd, then $Z \in \mathfrak{J}^*$, whilst $Z \in \mathfrak{J}^*$ if j is even.

LEMMA 5.1. Suppose that Ξ has a connected component Ξ^* which is of one of the types II5, III or IV. Then every X in $\mathfrak{J} - \mathfrak{J}^*$ is opposite each member of \mathfrak{J}^* .

PROOF. Assume false, and choose $X' \in \mathfrak{B}^*$ with $d(X, X') = d \neq 8$. Thus $d = 4$ or 6 because $X \notin \mathfrak{B}^*$ and Ξ^* is a connected component. Set $\langle X, X' \rangle = (X = X_1, X_2, \dots, X_{d+1} = X')$. If $d = 6$, then extend $\langle X, X' \rangle$ to a root by adjoining a path of length 2 in Ξ^* . Then Lemma 2.4 shows that $X_3 \in \mathfrak{B}$. Hence the connected component containing X always has a vertex Z satisfying $d(Z, X') = 4$. Now extend $\langle Z, X' \rangle$ to a root by adjoining a path of length 4 in Ξ^* . This is possible because of the assumption on the type of Ξ^* . Then Lemma 2.4 implies that $Z \in \mathfrak{B}^*$, this contradicts $X \notin \mathfrak{B}^*$.

We turn now to the proof of Theorem B. From now on, all statements concerning the structure of Ξ are made subject to the following assumption.

HYPOTHESIS 5.2. *The graph Ξ is disconnected and contains a non-trivial connected component which does not consist of a single vertex.*

If statement Y in Theorem B holds, then we say that Ξ is of type Y . Theorem B follows from Lemma 5.3 and Theorem 5.4 below.

LEMMA 5.3. *If Ξ has a non-trivial connected component which is invariant under E , then Ξ is one of the types VI1, VII ($i = 1$ or 2) or VIIi ($i = 1, 2$ or 3).*

PROOF. Let Ξ^* be a non-trivial connected component of Ξ which is fixed by E . Then Ξ^* is of type IIj with $j = 1, 2, 3$ or 4 . For suppose that this is not the case (i.e. assume that Ξ^* is one of the types II5, III or IV). Choose $X \in \mathfrak{B} - \mathfrak{B}^*$, let $Y \in \mathfrak{B}^*$, and let $\alpha \in G(X)$ with $\alpha \neq 1$. Then $d(X, Y) = 7$ by Lemma 5.1. Set $\langle X, Y \rangle = (X = X_1, X_2, \dots, X_8 = Y)$. Then X_5 belongs to $\langle Y, Y^\alpha \rangle$. Therefore $X_5 \in \mathfrak{B}^*$ because $Y, Y^\alpha \in \mathfrak{B}^*$ and Ξ^* is convex. But now Lemma 5.1 is contradicted because $d(X, X_5) = 4$.

Let Z be the centre of Ξ^* . Clearly Z must be fixed by E . Therefore Ξ is contained in $\Delta_{[4]}(Z)$, and in $\Delta_{[3]}(Z)$ if j is even. Now Lemma 5.3 follows easily.

THEOREM 5.4. *Assume that no non-trivial connected component of Ξ is left invariant by E . Then Ξ is one of the types V2, VI2, VIIi ($i = 2$ or 3) or VIII.*

PROOF. Let Ξ^* be a non-trivial connected component of Ξ . We consider the possibilities for Ξ^* .

LEMMA 5.5. *The component Ξ^* is neither of type III, nor of type IV.*

PROOF. Assume the statement to be false, and let \mathbb{F}_q be the field over which Ξ^* is defined. So q is a power of 2, and $G(X)$ is elementary abelian of order q for each $X \in \mathfrak{B}^*$. Choose $X_1 \in \mathfrak{B}^*$ and $X_2 \in \mathfrak{B}^*$ with X_1 and X_2 adjacent. Let $X' \in X_1^E$ lie in a connected component of Ξ which is distinct from Ξ^* . From Lemma 5.1, it follows that X_1 is opposite X' , and the only vertices in the root $\Phi = (X_1, X_2, \dots, X_9 = X')$ which belong to $\mathfrak{B} \cup \mathfrak{B}^*$ are X_1, X_2, X_8 and X_9 . Let $\Sigma = (X_0, X_1, \dots, X_{15})$ be an apartment which contains Φ .

Suppose that $q \neq 2$, and set $S = \langle G(X_1), G(X_9) \rangle$. Then $S \cong L_2(q)$ or $Sz(q)$ by Lemma 2.3. Let $Z = N_S(G(X_1)) \cap N_S(G(X_9))$, and let $D = N_S(Z)$. So $Z \cong Z_{q-1}$, and Z fixes each vertex in Φ (in particular, X_2), and $D \cong D_{2(q-1)}$ with each involution in D belonging to I and having centre in $\Delta_4(X_5) \cap \Delta_4(X_{13}) - \{X_1, X_9\}$. As each $z \in Z$ (with $z \neq 1$) can be written as a product of two distinct involutions in D , it follows from Lemma 1.5(ii) that Z is semi-regular on $\Delta_1(X_2) - \{X_1, X_3\}$. Therefore Z is semi-regular on $\Xi_1^*(X_2) - \{X_1\}$.

But this is impossible because the valency in Ξ^* of X_2 is either 2 or $q + 1$, according as Ξ^* is of type III or type IV.

Now suppose that $q = 2$. Then $M(X_2) \cong Z_2 \times Z_2 \cong M(X_8)$. Set $S = \langle M(X_2), M(X_8) \rangle$ and let Ω be the orbit of X_2 under S . So $X_8 \in \Omega$ and $\Omega \subseteq \Delta_3(X_5)$. Since only the extremities of $\langle X_2, X_8 \rangle$ belong to $\mathfrak{Z} \cup \mathfrak{Z}$, it follows that $d(Y, Y') = 6$ for every distinct $Y, Y' \in \Omega$. This implies that $N_S(M(X_2)) \cap M(X_2)^x = \{1\}$ for all $x \in S - N_S(M(X_2))$. Therefore $M(X_2) \cong Z_2 \times Z_2$ and Hering's theorem [5] imply that $S \cong L_2(4)$, and (Ω, S) is the natural 2-transitive group space. We may assume that $\langle G(X_1), G(X_9) \rangle \cong D_{10}$. Therefore $\langle G(X_1), G(X_9) \rangle$ is transitive on Ω ; in particular, $\Omega \subseteq \Delta_5(X_{13})$. Choose $X \in \Xi_1^*(X_2) - \{X_1\}$ and $\alpha \in G(X)$ with $\alpha \neq 1$. Then $X_8^\alpha \in \Delta_5(X_{13}) \cap \Delta_5(X_{13}^\alpha)$ because $\Omega^\alpha = \Omega$. Therefore $d(X_8^\alpha, X_{15}) = 3$ because $d(X_{13}, X_{13}^\alpha) = 4$ and $X_{15} \langle X_{13}, X_{13}^\alpha \rangle$. Hence $X_8^\alpha = X_2$ because $d(X_8^\alpha, X_5) = 3$. But this is absurd because X_2 is fixed by α .

LEMMA 5.6. *The component Ξ^* is not of type II5.*

PROOF. Assume to the contrary. Let (X_1, X_2, \dots, X_5) be a path of length 4 in Ξ^* , with X_5 the centre of Ξ^* . Choose $X \in X_1^E$ so that $X \notin \mathfrak{Z}^*$. Then $G(X_1)$, $G(X_3)$ and $G(X)$ are elementary abelian 2-groups by Lemma 3.3, and X is opposite to both X_1 and X_3 by Lemma 5.1. From Lemma 2.3 it follows that there exists $x \in \langle G(X_1), G(X_3), G(X) \rangle$ with $X_3 = X_1^x$. But this is impossible because x must leave Ξ^* invariant and yet move its centre X_5 .

LEMMA 5.7. *If Ξ^* is of type II4, then Ξ is of type V2.*

PROOF. Assume that Ξ^* is of type II4, and let Z be its centre (so $Z \in \mathfrak{Z}^*$). Since $Z^E \neq \{Z\}$, it is clearly sufficient to show that $d(Z, Z') = 2$ for every $Z' \in Z^E - \{Z\}$.

First, we argue that $G(X)$ is an elementary abelian 2-group for every $X \in \mathfrak{Z}^*$. If X is an extreme vertex, then this is certainly the case by Lemma 2.2(i). So let X be adjacent to Z , and assume that $G(X)$ is not an elementary abelian 2-group. Let (X_1, X_2, \dots, X_5) be a path of length 4 in Ξ^* , with $X = X_3$ and both X_1 and X_5 end vertices. Choose $X' \in X_1^E$ with $X' \notin \mathfrak{Z}^*$. Then $G(X)$ not an elementary abelian 2-group together with Lemmas 2.2(i) and 2.4 imply that X' is opposite to X , X_1 and X_5 . Now $G(X_1)$, $G(X_5)$ and $G(X')$ are all elementary abelian 2-groups, because X_1 and X_5 are extreme vertices of Ξ^* and $X' \in X_1^E$. By Lemma 2.3, there exist $\alpha, \alpha' \in I$ such that α (resp. α') interchanges X' and X_1 (resp. X_5), and both α and α' have centres opposite X . Hence $X_1^{\alpha\alpha'} = X_5$, and so $\alpha\alpha'$ leaves Ξ^* invariant. Therefore $\alpha\alpha'$ fixes Z , and hence $\alpha\alpha'$ fixes X . But X is opposite the centres of α and α' . Therefore $\alpha\alpha'$ fixes each vertex in $\Delta_1(X)$ by Lemma 1.5, and so $X_1^{\alpha\alpha'} \neq X_5$. This contradiction proves that $G(X)$ is an elementary abelian 2-group after all.

Let $Z' \in Z^E$, with $Z' \neq Z$. Then Z' is at distance 5 from each extreme vertex in Ξ^* . To see this, let X be extreme in \mathfrak{Z}^* , and choose X' to be adjacent to Z' in the connected component of Ξ which has Z' as centre. Then X and X' are in distinct orbits under E . Therefore $G(X)$ and $G(X')$ elementary abelian 2-groups, Lemma 2.3 and $X' \notin \mathfrak{Z}^*$ imply that $d(X, X') = 6$. Since there are at least two choices for X' , it follows that $d(X, Z') = 5$. Now let (X_1, X_2, \dots, X_7) be a path of length 7 in Ξ^* (so $Z = X_4$). Then Z' at distance 5 from each vertex in $X_1^{G(X_7)}$ implies that $d(X_3, Z') = 3$. Similarly, $d(X_5, Z') = 3$. Therefore $d(Z, Z') = 2$ as required.

LEMMA 5.8. *If Ξ^* is of type II3, then Ξ is of type VI2.*

PROOF. Assume that Ξ^* is of type II3, and let Z be its centre (so $Z \in \mathfrak{Z}^*$). Clearly, it suffices to show that $d(Z, Z') = 4$ for all $Z' \in Z^E - \{Z\}$.

As in Lemma 5.7, we first argue that $G(X)$ is an elementary abelian 2-group for every $X \in \mathfrak{B}^*$. Once we have proved this for each end vertex (i.e. each element of $\mathfrak{B}^* - \{Z\}$), then the argument used in the proof of Lemma 5.7 shows that it is also true for Z . So let $X \in \mathfrak{B}^* - \{Z\}$, and assume that $G(X)$ is not an elementary abelian 2-group. Choose $X' \in X^E$ with $X' \notin \mathfrak{B}^*$. Then Lemma 2.2 may be applied to show that X' is opposite each member of \mathfrak{B}^* . Let Ω be the orbit of X under $\langle G(X), G(X') \rangle$. Then the elements of Ω are pairwise opposites, and each member of $\Omega - \{X\}$ is opposite to Z . Moreover, $X' \in \Omega$. As $|G(X)| \geq 3$, we can find $\alpha \in G(U)$ and $\alpha' \in G(U')$, with $U, U' \in \Omega - \{X\}$ and $U \neq U'$, such that $X^{\alpha\alpha'} = X$. Therefore Ξ^* is invariant under $\alpha\alpha'$, and so $X^{\alpha\alpha'} = Z$. But Z is opposite to U and U' , and $d(X, Z) = 2$, which is impossible by Lemma 1.5. Hence $G(X)$ is an elementary abelian 2-group.

Let $Z' \in Z^E$ with $Z' \neq Z$. Let X be an end vertex of Ξ^* . Then X and Z' are in distinct orbits under E . Hence $G(X)$ and $G(Z')$ elementary abelian 2-groups, Lemma 2.3 and $Z' \notin \mathfrak{B}^*$ imply that $d(X, Z') = 6$. Therefore Z' is at distance 6 from every extreme vertex in Ξ^* . But now Lemma 3.4 may be used to show that $d(Z, Z') = 4$.

COMPLETION OF THE PROOF OF THEOREM 5.4. In view of Lemmas 5.5–5.8, it may be assumed that each non-trivial connected component of Ξ is of type II1 or type II2. Moreover, one of the latter type exists by Hypothesis 5.2.

Let Ξ^* be a connected component of Ξ of type II2, and let Z be its centre (so $Z \in \mathfrak{B}^*$). Let Ω be the orbit of Z under E . We argue that $d(Z, Z') \leq 6$ for every $Z' \in \Omega$. For suppose that Z is opposite some element Z' in Ω . Write $Z = X_0, Z' = X_8$, choose X_1 in \mathfrak{B}^* , and form the root (X_0, X_1, \dots, X_8) . Let $X' \in X_1^E$ be adjacent to X_8 . Then $X' = X_7$ because otherwise Lemma 1.4 implies that $X_2 \in \mathfrak{B}$, and so Ξ^* of type II2 is contradicted. Now choose $X \in \Xi_1^* (X_0) - \{X_1\}$. Then Lemma 2.4 applied to the root $(X, X_0, X_1, \dots, X_7)$ implies that $X_5 \in \mathfrak{B}$. Therefore the connected component with centre $Z' (= X_8)$ is not of type II2. This is impossible because $Z' \in Z^E$.

If $d(Z, Z') \leq 4$ for all $Z' \in \Omega$, then Ξ is of type VIIi with $i = 2$ or 3 . So assume that $d(Z, Z') = 6$ for some $Z' \in \Omega$, and set $\langle Z, Z' \rangle = (Z = X_0, X_1, \dots, X_6 = Z')$. If $\Omega \subseteq \Delta_{[3]}(X_3)$, then Ξ is again of type VII3. So we may assume that $W \in \Omega$ exists with $d(W, X_3) \geq 5$. Therefore Ξ is of convex type (E_4) or convex type (E_8) . However, Ξ of type (E_4) implies that $X_1, X_5 \in \mathfrak{B}^*$, and this contradicts Ξ^* is of type II2 because $G(X_5)$ fixes X_1 but not X_0 . Therefore Ξ is of convex type (E_8) , and hence Ξ is of type VIII.

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